

Generalised method of moments estimation of mediation models and structural mean models

Tom Palmer

May 2013

THE UNIVERSITY OF
WARWICK

Warwick
Medical School

- ▶ Introduction to GMM
- ▶ I. Mediation models
 - ▶ Joint estimation of mediator and outcome models - delta method SE
 - ▶ Example
- ▶ II. Structural mean models
 - ▶ Multiplicative SMM
 - ▶ Logistic SMM
- ▶ Summary

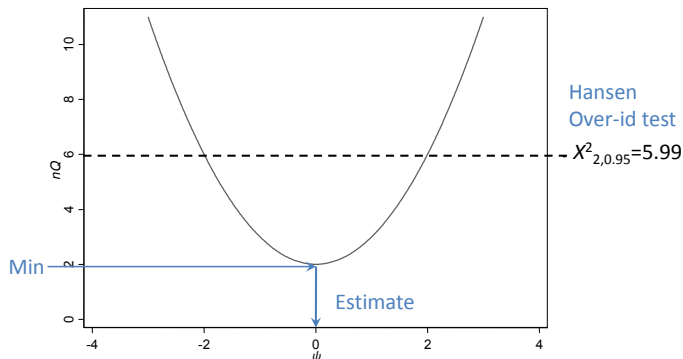
Introduction to Generalised Method of Moments (GMM) I

- ▶ Jointly solve system of moment conditions (equations)
- ▶ System: exactly identified, # instruments = # parameters
- ▶ System: over-identified, # instruments > # parameters
- ▶ m vector of moment conditions
- ▶ Minimises quadratic form w.r.t parameters (ψ)

$$Q = m'W^{-1}m = \left(\frac{1}{n} \sum_{i=1}^n m_i(\psi) \right)' W^{-1} \left(\frac{1}{n} \sum_{i=1}^n m_i(\psi) \right)$$

Introduction to Generalised Method of Moments (GMM) II

- ▶ Profiling over parameter of interest



- ▶ Over-identification test Hansen 1982: $nQ \sim \chi^2_q$
- ▶ In quadratic form: W affects efficiency (SEs) rather than consistency

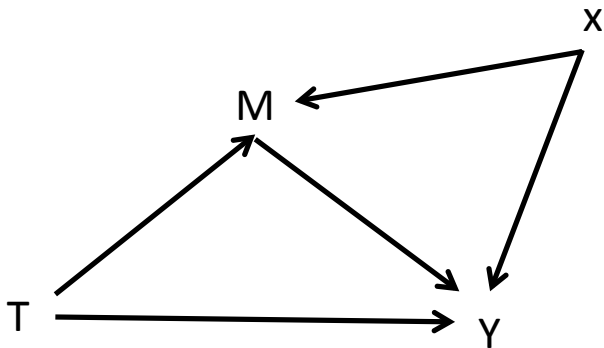
I. Mediation models

Mediation models – common implementation

Baron and Kenny (1986) type models

- ▶ Implementations:
 - ▶ Stata - medeff, medsens, paramed
 - ▶ R - mediation
 - ▶ bootstrapping for SEs of mediation parameters
- ▶ Fit mediator model
- ▶ Fit outcome model
- ▶ Mediation parameters – function of estimated parameters
- ▶ problem: fit models separately – missing covariance between parameters from different models

Mediation model with a single confounder



$$M = \alpha_0 + \alpha_1 T + \alpha_2 X + \epsilon_1$$

$$Y = \beta_0 + \beta_1 T + \beta_2 M + \beta_3 X + \epsilon_2$$

$$\hat{\mathbf{V}} = \begin{bmatrix} \hat{\sigma}_{\alpha 00}^2 & \hat{\sigma}_{\alpha 01} & \hat{\sigma}_{\alpha 02} & \cdot & \cdot & \cdot & \cdot \\ \hat{\sigma}_{\alpha 10} & \hat{\sigma}_{\alpha 11}^2 & \hat{\sigma}_{\alpha 12} & \cdot & \cdot & \cdot & \cdot \\ \hat{\sigma}_{\alpha 20} & \hat{\sigma}_{\alpha 21} & \hat{\sigma}_{\alpha 22}^2 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \hat{\sigma}_{\beta 00}^2 & \hat{\sigma}_{\beta 01} & \hat{\sigma}_{\beta 02} & \hat{\sigma}_{\beta 03} \\ \cdot & \cdot & \cdot & \hat{\sigma}_{\beta 10} & \hat{\sigma}_{\beta 11}^2 & \hat{\sigma}_{\beta 12} & \hat{\sigma}_{\beta 13} \\ \cdot & \cdot & \cdot & \hat{\sigma}_{\beta 20} & \hat{\sigma}_{\beta 21} & \hat{\sigma}_{\beta 22}^2 & \hat{\sigma}_{\beta 23} \\ \cdot & \cdot & \cdot & \hat{\sigma}_{\beta 30} & \hat{\sigma}_{\beta 31} & \hat{\sigma}_{\beta 32} & \hat{\sigma}_{\beta 33}^2 \end{bmatrix}$$

Imai et al. (2010)

Natural indirect effect = $\beta_2\alpha_1$

$$V(\text{NIE}) = \alpha_1^2\sigma_{\beta_2}^2 + \beta_2^2\sigma_{\alpha_1}^2 + 2\beta_2\alpha_1\text{cov}(\beta_2, \alpha_1)$$

- ▶ bootstrap to estimate SE of mediation parameter
- ▶ GMM – *joint estimation of mediation and outcome models*
 - ▶ full var-covar matrix then delta-method SE

GMM estimation of GLMs

Estimating equation for GLMs - solve wrt β :

$$\sum_{i=1}^n x_i(y_i - g^{-1}(X\beta)) = 0$$

Model	Link fn $g(\mu)$	Inverse link $g^{-1}(X\beta)$
Linear	$I(\mu)$	$I(X\beta)$
Poisson	$\log(\mu)$	$\exp(X\beta)$
Logistic	$\text{logit}(\mu) = \log\left(\frac{\mu}{1-\mu}\right)$	$\text{expit}(X\beta) = \frac{\exp(X\beta)}{1+\exp(X\beta)}$
Probit	$\Phi^{-1}(\mu)$	$\Phi(X\beta)$

In (exactly identified) GMM use covariates as *instruments* for themselves

Moment conditions

$$E[(M - \alpha_0 - \alpha_1 T - \alpha_2 x)\mathbf{1}] = 0$$

$$E[(M - \alpha_0 - \alpha_1 T - \alpha_2 x)T] = 0$$

$$E[(M - \alpha_0 - \alpha_1 T - \alpha_2 x)x] = 0$$

$$E[(M - \beta_0 - \beta_1 T - \beta_2 M - \beta_3 x)\mathbf{1}] = 0$$

$$E[(M - \beta_0 - \beta_1 T - \beta_2 M - \beta_3 x)T] = 0$$

$$E[(M - \beta_0 - \beta_1 T - \beta_2 M - \beta_3 x)M] = 0$$

$$E[(M - \beta_0 - \beta_1 T - \beta_2 M - \beta_3 x)x] = 0$$

Instruments

Implementation

- ▶ Stata: `gmm` then `lincom/nlcom`
- ▶ R: `gmm` package then `deltamethod()` from `MSM` package

Example from medsens helpfile I

$$M = \alpha_0 + \alpha_1 T + \alpha_2 X + \epsilon_1$$

$$Y = \beta_0 + \beta_1 T + \beta_2 M + \beta_3 X + \epsilon_2$$

	Estimate (95% CI)
α_0	0.28 (0.19, 0.36)
α_1	0.17 (0.04, 0.29)
α_2	0.27 (0.21, 0.33)
β_0	0.32 (0.24, 0.41)
β_1	-0.58 (-0.70, -0.46)
β_2	0.71 (0.65, 0.77)
β_3	0.27 (0.21, 0.34)

Estimated variance-covariance matrix:

0.00178							
-0.00178	0.00398						
-0.00001	0.00008	0.0009					
0.00009	-0.00009	0.00005	0.00193				
-0.00007	-0.00003	-0.000002	-0.00181	0.0037			
-0.00005	0.00007	-0.00002	-0.0003	-0.00007	0.0009		
0.00006	-0.00004	-0.00003	0.0002	-0.00009	-0.0002	0.0010	

Example from medsens helpfile II

Estimates of mediation parameters

	Estimate	95% CI	
		Bootstrap	DM (using GMM)
NIE = $\beta_2\alpha_1$	0.12	(0.035, 0.211)	(0.031, 0.209)
CDE = β_1	-0.58	(-0.697, -0.451)	(-0.697, -0.457)
Total effect	-0.46	(-0.610, -0.304)	(-0.605, -0.309)
% TE mediated	-0.26	(-0.397, -0.198)	(-0.517, -0.008)

DM: delta-method

SEs from joint maximum likelihood estimation

$$\log\text{Like} = \log\text{Like}_{\text{Mediator}} + \log\text{Like}_{\text{Outcome}}$$

	Estimate	Bootstrap	95% CI	
			DM (GMM)	DM (ML)
NIE = $\beta_2\alpha_1$	0.12	(0.035, 0.211)	(0.031, 0.209)	(0.031, 0.208)
CDE = β_1	-0.58	(-0.697, -0.451)	(-0.697, -0.457)	(-0.697, -0.457)
Total effect	-0.46	(-0.610, -0.304)	(-0.605, -0.309)	(-0.605, -0.308)
% TE mediated	-0.26	(-0.397, -0.198)	(-0.517, -0.008)	(-0.515, -0.009)

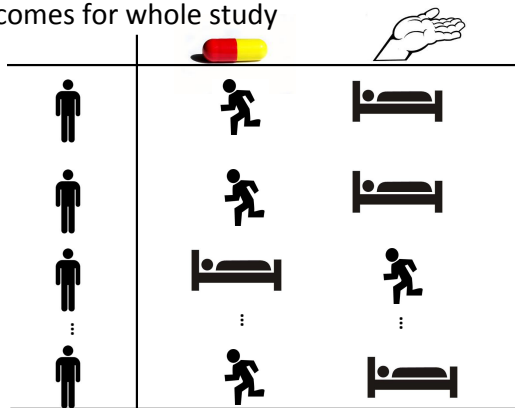
DM: delta-method

GMM SEs here are heteroskedasticity robust SEs

II. Structural mean models

Structural mean models

Potential outcomes for whole study



Average treatment effect = $E[Y(X=1)] - E[Y(X=0)]$
binary outcome: causal risk difference

Causal risk ratio = $E[Y(X=1)] / E[Y(X=0)]$

Causal odds ratio = $\text{odds}[Y(X=1)] / \text{odds}[Y(X=0)]$

Multiplicative SMM

- ▶ X exposure/treatment
- ▶ Y outcome
- ▶ Z instrument
- ▶ $Y(X = 0)$ exposure/treatment free potential outcome

Hernan & Robins 2006

$$\frac{E[Y|X, Z]}{E[Y(0)|X, Z]} = \exp(\psi X)$$

ψ : log causal risk ratio

Rearrange for $Y(0)$: $Y(0) = Y \exp(-\psi X)$

Multiplicative SMM: estimation with multiple instruments

Under the instrumental variable assumptions Robins 1994:

$$Y(0) \perp\!\!\!\perp Z$$
$$Y \exp(-\psi X) \perp\!\!\!\perp Z$$

Multiplicative SMM: estimation with multiple instruments

Under the instrumental variable assumptions Robins 1994:

$$Y(0) \perp\!\!\!\perp Z$$

$$Y \exp(-\psi X) \perp\!\!\!\perp Z$$

trick: $Y \exp(-\psi X) - Y(0) \perp\!\!\!\perp Z$

Multiplicative SMM: estimation with multiple instruments

Under the instrumental variable assumptions Robins 1994:

$$Y(0) \perp\!\!\!\perp Z$$

$$Y \exp(-\psi X) \perp\!\!\!\perp Z$$

trick: $Y \exp(-\psi X) - Y(0) \perp\!\!\!\perp Z$

Moment conditions

$Z=0,1$

$$E[(Y \exp(-\psi X) - Y(0))1] = 0$$

$$E[(Y \exp(-\psi X) - Y(0))Z_1] = 0$$

Multiplicative SMM: estimation with multiple instruments

Under the instrumental variable assumptions Robins 1994:

$$Y(0) \perp\!\!\!\perp Z$$

$$Y \exp(-\psi X) \perp\!\!\!\perp Z$$

trick: $Y \exp(-\psi X) - Y(0) \perp\!\!\!\perp Z$

Moment conditions

$Z=0,1,2,3$

Over-identified

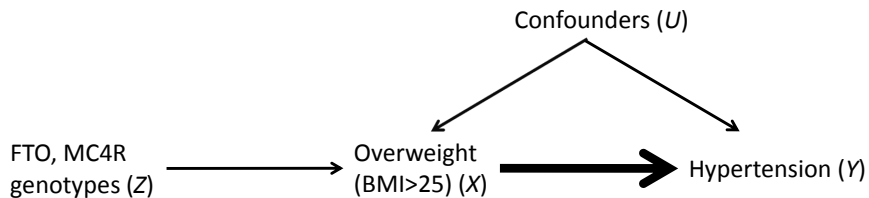
$$E[(Y \exp(-\psi X) - Y(0))1] = 0$$

$$E[(Y \exp(-\psi X) - Y(0))Z_1] = 0$$

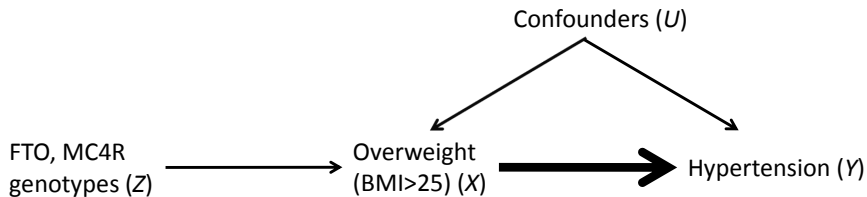
$$E[(Y \exp(-\psi X) - Y(0))Z_2] = 0$$

$$E[(Y \exp(-\psi X) - Y(0))Z_3] = 0$$

Copenhagen example descriptive statistics 1



Copenhagen example descriptive statistics 1



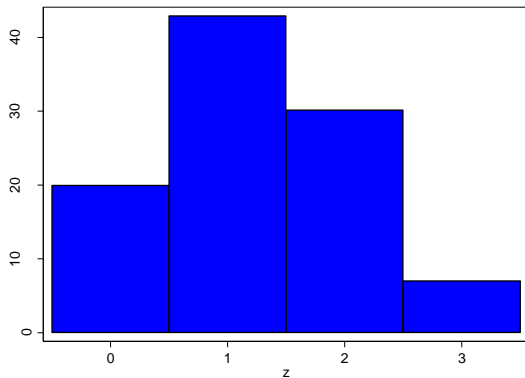
	No Hypertension	Hypertension	Total
Not Overweight	10,066 42%	13,909 58%	23,975
Overweight	6,906 22%	24,642 78%	31,548
Total	16,972 31%	38,551 69%	55,523 $\chi^2 P < 0.001$

Risk ratio for hypertension 1.35 (1.32, 1.37)

Copenhagen example descriptive statistics 2

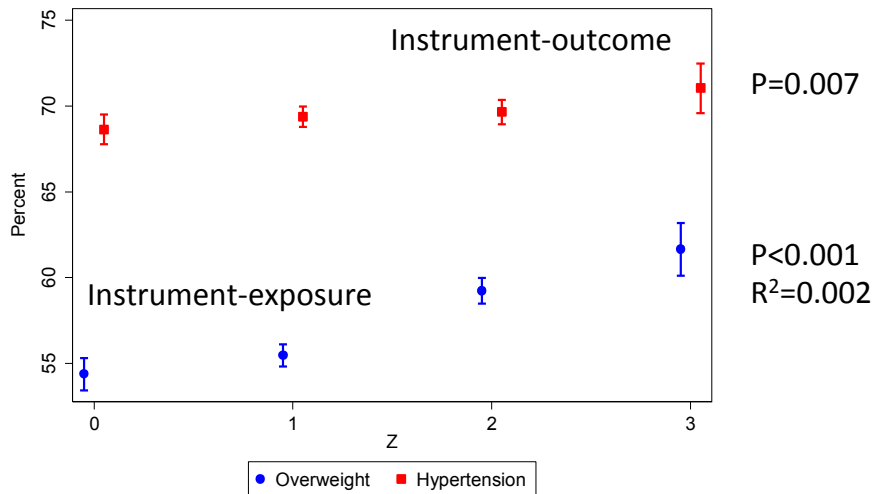
Distribution of instrument (Z)

<i>FTO</i>	<i>MC4R</i>	Z	Freq
0	0	0	0.20
0	1	1	0.15
1	0	1	0.27
1	1	2	0.21
2	0	2	0.09
2	1	3	0.07

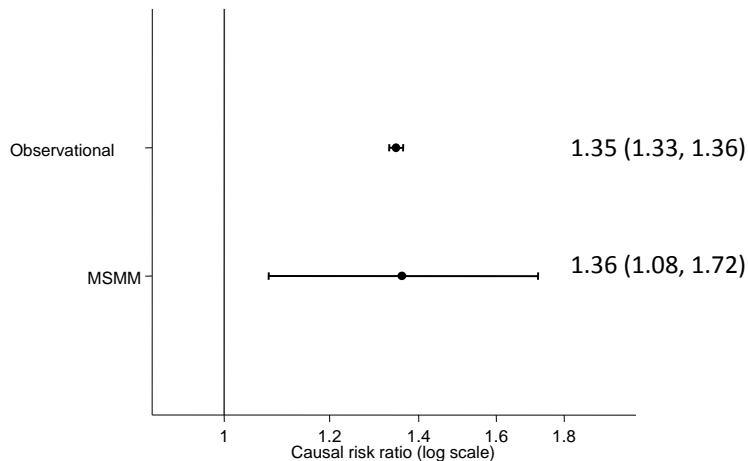


Copenhagen example descriptive statistics 3

Exposure (over-weight) & outcome (hypertension) by instrument



Copenhagen example Multiplicative SMM estimates



MSMM: Hansen over-identification test $P = 0.31$

$E[Y(0)] = 0.58 (0.50, 0.65)$

How does GMM deal with multiple instruments?

GMM estimator solution to:

$$\frac{\partial m'(\psi)}{\partial \psi} W^{-1} m(\psi) = 0$$

- ▶ MSMM: instruments combined into linear projection of $YX \exp(-X\psi)$ on $Z = (1, Z_1, Z_2)'$ Bowden & Vansteelandt 2010

(double) Logistic SMM

$$\text{logit}(p) = \log(p/(1 - p)), \text{expit}(x) = e^x/(1 + e^x)$$

Goetghebeur, 2010

$$\text{logit}(E[Y|X, Z]) - \text{logit}(E[Y(0)|X, Z]) = \psi X$$

ψ : log causal odds ratio

Rearrange for $Y(0)$: $Y(0) = \text{expit}(\text{logit}(Y) - \psi X)$

(double) Logistic SMM

$$\text{logit}(p) = \log(p/(1 - p)), \text{expit}(x) = e^x/(1 + e^x)$$

Goetghebeur, 2010

$$\text{logit}(E[Y|X, Z]) - \text{logit}(E[Y(0)|X, Z]) = \psi X$$

ψ : log causal odds ratio

Rearrange for $Y(0)$: $Y(0) = \text{expit}(\text{logit}(Y) - \psi X)$

- ▶ Can't be estimated in a single step Robins (1999)
- ▶ First stage association model Vansteelandt (2003):
 - (i) logistic regression of Y on X & Z & interactions
 - (ii) predict Y , estimate LSMM using predicted Y

(double) Logistic SMM moment conditions

Association model moment conditions

Logistic regression using GMM

$$E[(Y - \text{expit}(\beta_0 + \beta_1 X + \beta_2 Z + \beta_3 XZ))1] = 0$$

$$E[(Y - \text{expit}(\beta_0 + \beta_1 X + \beta_2 Z + \beta_3 XZ))X] = 0$$

$$E[(Y - \text{expit}(\beta_0 + \beta_1 X + \beta_2 Z + \beta_3 XZ))Z] = 0$$

$$E[(Y - \text{expit}(\beta_0 + \beta_1 X + \beta_2 Z + \beta_3 XZ))XZ] = 0$$

(double) Logistic SMM moment conditions

Association model moment conditions

Logistic regression using GMM

$$E[(Y - \text{expit}(\beta_0 + \beta_1 X + \beta_2 Z + \beta_3 XZ))1] = 0$$

$$E[(Y - \text{expit}(\beta_0 + \beta_1 X + \beta_2 Z + \beta_3 XZ))X] = 0$$

$$E[(Y - \text{expit}(\beta_0 + \beta_1 X + \beta_2 Z + \beta_3 XZ))Z] = 0$$

$$E[(Y - \text{expit}(\beta_0 + \beta_1 X + \beta_2 Z + \beta_3 XZ))XZ] = 0$$

Causal model moment conditions

$$E[(\text{expit}(\text{logit}(\hat{p}) - \psi X) - Y(0))1] = 0$$

$$E[(\text{expit}(\text{logit}(\hat{p}) - \psi X) - Y(0))Z] = 0$$

Problem: SEs incorrect - need association model uncertainty

LSMM joint estimation

Joint estimation = correct SEs [Gourieroux \(1996\)](#)

[Vansteelandt & Goetghebeur \(2003\)](#)

$$E[(Y - \text{expit}(\beta_0 + \beta_1 X + \beta_2 Z + \beta_3 XZ))1] = 0$$

$$E[(Y - \text{expit}(\beta_0 + \beta_1 X + \beta_2 Z + \beta_3 XZ))X] = 0$$

$$E[(Y - \text{expit}(\beta_0 + \beta_1 X + \beta_2 Z + \beta_3 XZ))Z] = 0$$

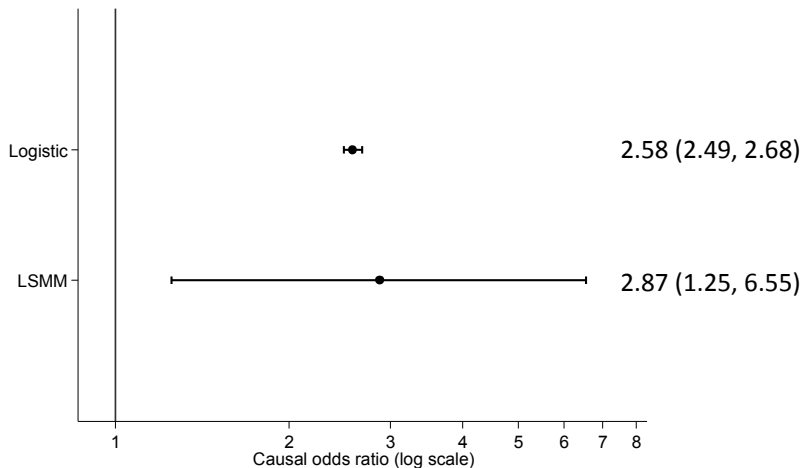
$$E[(Y - \text{expit}(\beta_0 + \beta_1 X + \beta_2 Z + \beta_3 XZ))XZ] = 0$$

$$E[(\text{expit}(\beta_0 + \beta_1 X + \beta_2 Z + \beta_3 XZ) - \psi X)1] = 0$$

$$E[(\text{expit}(\beta_0 + \beta_1 X + \beta_2 Z + \beta_3 XZ) - \psi X)Z] = 0$$

In example causal model SEs increase $\times 10$ from non-joint estimation

Copenhagen example LSMM estimates



LSMM: Hansen over-identification test $P = 0.29$

$E[Y(0)] = 0.57$ (0.45, 0.68)

Issues estimating SMMs

- ▶ Weak identification
 - ▶ many values of causal parameter give independence condition close to zero
- ▶ GMM convergence at local/global minima
- ▶ Hence check estimated $E[Y(0)]$ approx baseline risk
- ▶ Sensitive to initial values: in another dataset
 - ▶ initial $CRR = 1$ gave $CRR > 1$
 - ▶ initial $CRR < 1$ gave $CRR < 1$
- ▶ Fit with centred Z (with/without constant $E[Y(0)]$)
- ▶ Estimation MSMM/LSMM models with continuous X more problematic than binary X – centring X important for sensible estimates of $E[Y(0)]$

Summary

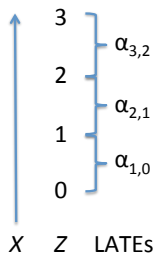
- ▶ Mediation models
 - ▶ GMM exact identification
 - ▶ Delta-method SEs alternative to bootstrapping
- ▶ Structural mean models
 - ▶ fit over-identified models with multiple instruments
 - ▶ check if estimated $E[Y(0)]$ is sensible – approx baseline risk
- ▶ Straightforward to implement in Stata & R

References

- ▶ Baron RM, Kenny DA (1986). The Moderator-Mediator Variable Distinction in Social Psychological Research: Conceptual, Strategic, and Statistical Considerations. *Journal of Personality and Social Psychology*, 51(6), 1173–1182.
- ▶ Hansen. Large sample properties of generalized method of moments estimators *Econometrica*, 1982, 50, 1029-1054.
- ▶ Imai, Keele, Yamamoto (2010) Identification, Inference, and Sensitivity Analysis for Causal Mediation Effects, *Statistical Sciences*, 25(1) pp. 51-71.
- ▶ Hicks, Raymond and Dustin Tingley (2011) mediation: Stata package for causal mediation analysis. (*The Stata Journal*, 11(4) 2011)
- ▶ Robins JM (1994) Correcting for non-compliance in randomized trials using structural nested mean models. *CSTM*. 23(8) 2379–2412

Local risk ratios for Multiplicative SMM

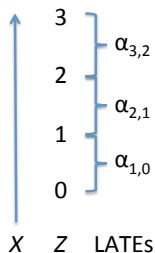
- ▶ Identification: NEM by Z ... what if it doesn't hold?
- ▶ Alternative assumption of monotonicity: $X(Z_k) \geq X(Z_{k-1})$
- ▶ Local Average Treatment Effect (LATE) Imbens 1994
 - ▶ effect among those whose exposures are changed (upwardly) by changing (counterfactually) the IV from Z_{k-1} to Z_k



$$\alpha_{\text{All}} = \lambda_1 \alpha_{1,0} + \lambda_2 \alpha_{2,1} + \lambda_3 \alpha_{3,2}$$

Local risk ratios for Multiplicative SMM

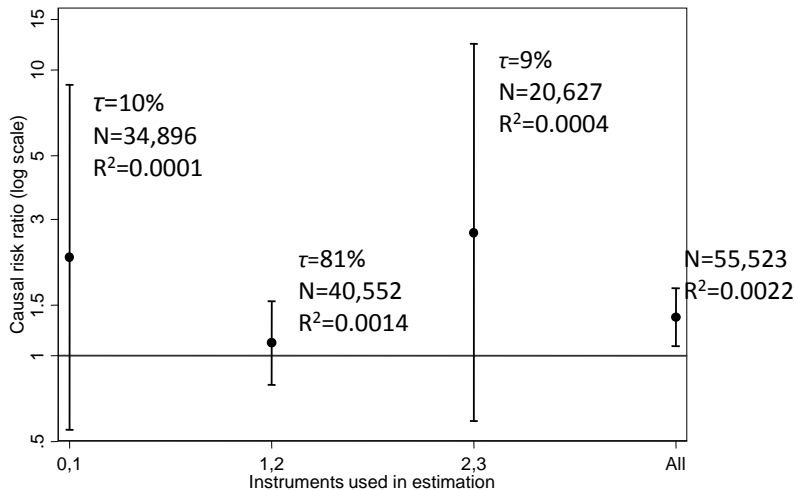
- ▶ Identification: NEM by Z ... what if it doesn't hold?
- ▶ Alternative assumption of monotonicity: $X(Z_k) \geq X(Z_{k-1})$
- ▶ Local Average Treatment Effect (LATE) Imbens 1994
 - ▶ effect among those whose exposures are changed (upwardly) by changing (counterfactually) the IV from Z_{k-1} to Z_k



$$\alpha_{\text{All}} = \lambda_1 \alpha_{1,0} + \lambda_2 \alpha_{2,1} + \lambda_3 \alpha_{3,2}$$

Similar result holds for MSMM:
$$e_{\text{All}}^{\psi} = \sum_{k=1}^K \tau_k e_{k,k-1}^{\psi}$$

Copenhagen example local risk ratios



$$\text{Check: } (0.10 \times 2.21) + (0.81 \times 1.11) + (0.09 \times 2.69) = 1.36$$